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# Middle convolution and the hypergeometric equation

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**Abstract**
The addition and the Detweiler and Reiter algebraic analogue of Katz's middle convolution are certain transformations of Fuchsian systems. Their compositions are useful to get nontrivial relations between solutions of these systems. The aim of this paper is to study the result of these transformations for a $2 \times 2$ linear Fuchsian system with triangular matrices and four singularities; the isomonodromy deformations of which are given by the hypergeometric functions.

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## 1. The hypergeometric equation and monodromy preserving deformations of Fuchsian systems

The hypergeometric equation

$$t\,(1-t)\,u'' + \{c - (a+b+1)\,t\}\,u' - a\,b\,u = 0, \tag{1}$$

where $' = \mathrm{d}/\mathrm{d}t$ and $a, b, c$ are arbitrary parameters, appears in many areas of pure and applied mathematics, theoretical physics and statistics. For instance, many orthogonal polynomials are defined with the help of the hypergeometric-type functions. The only singularities of the equation (and of the solutions, since the equation is linear) are $t = 0, 1$ and $\infty$. The famous Gauss hypergeometric function is a solution to (1). We refer the reader to [1], [6, vol I] for a basic theory of the hypergeometric functions.

Let us briefly recall the notion of isomonodromy deformations of linear systems following [11–13, 15].

Let

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi = A(\lambda)\Psi, \qquad A(\lambda) \in \mathrm{SL}(n, \mathbb{C}) \tag{2}$$

be a linear $n \times n$ system with the matrix given by

$$A(\lambda) = \sum_{k=1}^{N} \frac{A_k}{\lambda - t_k},$$

where $t_k$ are distinct points in $\mathbb{C}$ and the residue matrices $A_k$ do not depend on $\lambda$. Such systems are called *Fuchsian systems*.

Fixing $\lambda_0 = \infty$ and imposing the normalization condition for the fundamental solution $\Psi(\lambda_0) = I$ one can define the monodromy matrices $M_k \in \mathrm{SL}(n, \mathbb{C}), k = 1, \ldots, N, \infty$, of the analytic continuation of the fundamental solution $\Psi$ along the generators of the fundamental group $\pi_1(\mathbb{C}P^1 \backslash \{t_1, t_2, \ldots, t_N, \infty\}, \lambda_0)$. The monodromy matrices satisfy the relation $M_\infty M_N \cdots M_1 = I$. A group generated by the monodromy matrices $M_k$ is called a monodromy representation of the fundamental group or a monodromy group. The isomonodromy (or, equivalently, monodromy preserving) condition means that the matrices $M_k$ do not depend on the positions of the poles, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t_i} M_k = 0.$$

Under certain non-resonance assumptions on the eigenvalues $\theta_k$ of the matrices $A_k$ and $A_\infty := -\sum_{j=1}^N A_j$ one can show that the function $\Psi$ satisfies the following system:

$$\frac{\partial}{\partial t_i} \Psi = -\frac{A_i}{\lambda - t_i} \Psi, \qquad i = 1, \ldots, N, \tag{3}$$

in the case of the monodromy preserving deformations. The compatibility conditions of (2) and (3) are known as the Schlesinger equations, or deformation equations,

$$\frac{\partial A_k}{\partial t_i} = \frac{[A_i, A_k]}{t_i - t_k}, \qquad k \neq i,$$

$$\frac{\partial A_i}{\partial t_i} = -\sum_{k=1, k\neq i}^{N} \frac{[A_i, A_k]}{t_i - t_k}, \qquad k = i.$$

When $N = 3$, $t_1 = 0, t_2 = 1, t_3 = t$ and $n = 2$, this system is equivalent to the sixth Painlevé equation ($P_{VI}$). It is a nonlinear second-order differential equation whose solutions are meromorphic on the universal cover of $\mathbb{C}\backslash\{0, 1\}$ (for more information see [12, 18]).

Kitaev [15] introduced a notion of a special function of the isomonodromy type and showed that most of the special functions of applied mathematics and mathematical physics belong to this type. According to the definition of Kitaev, a function is called a *special function of the isomonodromy type* if it defines a solution of the Schlesinger system (3), which is the system of isomonodromy deformations of (2). In other words, when the function governs isomonodromy deformations of a system of linear ordinary differential equations with rational coefficients. Moreover, the function is called a *linear special function of isomonodromy type* if system (2) has a triangular matrix $A(\lambda)$, i.e. $A(\lambda)_{ij} = 0$ for $j > i$. This definition provides a unified approach to the theories of linear special functions (classical transcendental functions) and the nonlinear special functions (functions of the Painlevé type). Some of the properties of special functions can easily be derived via the isomonodromy approach [15].

The Gauss hypergeometric function is a linear special function of the isomonodromy type as shown below following [15]. Take $N = 3, t_1 = 0, t_2 = 1, t_3 = t$ and $n = 2$ in (2) and consider the system given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \Psi = \left( \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t} \right) \Psi, \tag{4}$$

with the triangular matrices

$$A_k = \begin{pmatrix} 0 & 0 \\ u_k(t) & 0 \end{pmatrix} + \theta_k \sigma_3.$$

In addition, let us assume $u_0 + u_1 + u_t = 0$. The Schlesinger equations give the following system for the functions $u_0, u_1, u_t$:

$$\frac{\mathrm{d}u_0}{\mathrm{d}t} = \frac{2\theta_0 u_t - 2\theta_t u_0}{t}, \qquad \frac{\mathrm{d}u_1}{\mathrm{d}t} = \frac{2\theta_1 u_t - 2\theta_t u_1}{t-1}, \tag{5}$$

which is equivalent to the Euler differential equation (1) for $u_0$ with

$$a = 2\theta_t, \quad b = 2\theta_0 + 2\theta_1 + 2\theta_t, \quad c = 2\theta_0 + 2\theta_t + 1.$$

Similarly, the functions $u_1(t)$ and $u_t(t)$ solve (1) with $a = 2\theta_t, b = 2\theta_0 + 2\theta_1 + 2\theta_t, c = 2\theta_0 + 2\theta_t$ and $a = 2\theta_t + 1, b = 2\theta_0 + 2\theta_1 + 2\theta_t, c = 2\theta_0 + 2\theta_t + 1$, respectively. Thus, the functions $u_k$ are the hypergeometric functions. As mentioned above when the matrices in (4) are not triangular, we have the nonlinear analogue of the hypergeometric function, the sixth Painlevé equation, depending on four parameters. It is known [18] that $(P_{VI})$ has solutions expressed in terms of the hypergeometric functions for certain non-generic values of its parameters.

## 2. Addition and middle convolution

Let $M(n, \mathbb{C})$ be the space of $n \times n$ complex matrices. Assume that we are given a tuple of matrices $\mathbf{A} = (A_1, \ldots, A_r)$, $A_k \in M(n, \mathbb{C})$. Let us also fix points $\lambda = t_k \in \mathbb{C}, k = 1, \ldots, r$, and consider a Fuchsian system given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_1 = \sum_{k=1}^{r} \frac{A_k}{\lambda - t_k}\Psi_1. \tag{6}$$

Denote the residue matrix at infinity by $A_\infty = -\sum_{k=1}^{r} A_k$.

The operation of *addition* is simply the change of the eigenvalues of the residue matrix: $A_k \to A_k + \alpha I_n$, where $\alpha \in \mathbb{C}$, $I_n$ is the identity matrix. Such a transformation is obtained by a gauge transformation $\Psi_1 \to (\lambda - t_k)^{-\alpha}\Psi_1$.

Further we discuss the Dettweiler and Reiter algebraic construction of Katz's middle convolution functor following [2–5].

By Katz's theory [14] any irreducible rigid local system on the punctured affine line can be obtained from a local system of rank 1 by applying a suitable sequence of middle convolutions and scalar multiplications. Dettweiler and Reiter presented a purely algebraic analogue of Katz's middle convolution functor. Their additive version of middle convolution depends on a scalar $\mu \in \mathbb{C}$ and is denoted by $mc_\mu$. It is a transformation on tuples of matrices

$$(A_1, \ldots, A_r) \in (M(n, \mathbb{C}))^r \to mc_\mu(A_1, \ldots, A_r) = (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in (M(m, \mathbb{C}))^r$$

constructed as follows. We review the algorithm in detail because of the necessity to refer to it in the following section.

First, for $\mu \in \mathbb{C}$ one defines *convolution matrices* $\mathbf{B} = c_\mu(\mathbf{A}) = (B_1, \ldots, B_r)$ by

$$B_k = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ A_1 & \cdots & A_{k-1} & A_k + \mu I_n & A_{k+1} & \cdots & A_r \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M(nr, \mathbb{C})$$

such that $B_k$ is zero outside the $k$th block row.

The convolution matrices define a new Fuchsian system with $nr$ equations and with the same number of singularities as in the original Fuchsian system:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_2 = \sum_{k=1}^{r} \frac{B_k}{\lambda - t_k}\Psi_2. \tag{7}$$

The relation between the convolution operation $c_\mu$ and the Euler integral transformation between solutions of systems (6) and (7) is described in [5, sections 4, 5] (see also [8, section 2] for a summary). System (7) may be reducible with the following invariant subspaces:

$$\mathcal{L} = \oplus_{k=1}^{r}\mathcal{L}_k, \qquad \mathcal{L}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Ker}(A_k) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \ (k\text{th entry}), \qquad k = 1, \ldots, r, \tag{8}$$

and

$$\mathcal{K} = \bigcap_{k=1}^{r} \mathrm{Ker}(B_k) = \mathrm{Ker}(B_1 + \cdots + B_r). \tag{9}$$

Let us fix an isomorphism between $\mathbb{C}^{nr}/(\mathcal{K}+\mathcal{L})$ and $\mathbb{C}^m$ for some $m$. The matrices $\widetilde{\mathbf{B}} = mc_\mu(\mathbf{A}) = (\widetilde{B}_1, \ldots, \widetilde{B}_r) \in M(m, \mathbb{C})$, where $\widetilde{B}_k$ is induced by the action of $B_k$ on $\mathbb{C}^m \simeq \mathbb{C}^{nr}/(\mathcal{K}+\mathcal{L})$, are called the middle convolution matrices of $\mathbf{A}$. Thus, the resulting irreducible Fuchsian system with $m$ equations is given by

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\Psi_3 = \sum_{k=1}^{r} \frac{\widetilde{B}_k}{\lambda - t_k}\Psi_3, \tag{10}$$

and this procedure is called *the additive version of the middle convolution with parameter* $\mu$.

The Dettweiler and Reiter algorithm can formally be applied to any Fuchsian system, not necessarily the rigid one. It guarantees that the resulting linear system is irreducible and has the same number of singularities as the initial Fuchsian system. Moreover, the deformation equations, if any, are preserved [10]. However, the dimension of matrices of the resulting Fuchsian system may change, and, hence, the number of equations of the Fuchsian system may be different. As mentioned above, the Fuchsian systems before and after middle convolution are related by the Euler integral transformation along a Pochhammer contour [5, section 4]. This algebraic interpretation of middle convolution proved to be useful for a number of problems. For instance, the integral transformation between solutions of the Heun equation can be rediscovered if middle convolution is applied to the hypergeometric system of the Heun equation as shown in [8]. In [7] the algorithm is applied to the general $2 \times 2$ Fuchsian system with four singularities. The deformation equation is then the sixth Painlevé equation with generic values of the parameters. The middle convolution transformation yields Okamoto's birational transformation for the deformation equation. Since the sixth Painlevé equation has solutions expressed in terms of the hypergeometric functions for certain non-generic values of the parameters [18], it is a natural question to ask what happens to system (4) under addition and middle convolution transformations. Thus, the main objective of the current paper is to study the compositions of addition and middle convolution transformations of system (4) in detail. This will contribute to the theory of integral transformations of differential equations and its connection to linear and nonlinear special functions (e.g. [8, 17, 19]).

### 3. Middle convolution and the hypergeometric equation

In this section we first apply different additions to system (4) and then study the effect of the middle convolution transformation. We are interested in the type of equations we get and, if the resulting system is also $2 \times 2$, what kind of relations we obtain for the hypergeometric functions.

**Case 1.** Shifting the eigenvalues of the residue matrices in (4) by addition $\Psi = \lambda^{-\theta_0}(\lambda - 1)^{-\theta_1}(\lambda - t)^{\theta_t}\Psi_1$ we start with the system (6) with

$$A_1 = \begin{pmatrix} 2\theta_0 & 0 \\ u_0(t) & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2\theta_1 & 0 \\ u_1(t) & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ u_t(t) & -2\theta_t \end{pmatrix}.$$

We apply the middle convolution algorithm with parameter $\mu$ not equal to the eigenvalues of the residue matrix at $\lambda = \infty$. We also assume that $\mu \neq 0$. This is a generic case. In this case the convolution matrices are of dimension $nr = 6$. The corresponding system (7) is reducible. The invariant subspace $\mathcal{K}$ in (9) is empty and the subspace $\mathcal{L}$ in (8) is spanned by three vectors in $\mathbb{C}^6$. The quotient space $\mathbb{C}^3 \simeq \mathbb{C}^6/(\mathcal{K} + \mathcal{L})$ is constructed by adding three more vectors to the basis of the invariant subspaces, for instance, vectors $e_1$, $e_3$ and $e_5$, where $e_k$ has 1 at position $k$ and other elements are equal to zero. In general, the result might be slightly different if we choose different vectors.

Thus, for generic values of the parameter $\mu$ in middle convolution we get $3 \times 3$ linear system (10) with

$$\widetilde{B}_1 = \begin{pmatrix} 2\theta_0 + \mu & 2\theta_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 2\theta_0 & 2\theta_1 + \mu & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\theta_0 - \frac{2\theta_t u_0(t)}{u_t(t)} & 2\theta_1 - \frac{2\theta_t u_1(t)}{u_t(t)} & -2\theta_t + \mu \end{pmatrix}.$$

The eigenvalues of the residue matrix of (10) at $\lambda = \infty$ are equal to $-2(\theta_0 + \theta_1) - \mu$, $2\theta_t - \mu$, $-\mu$. Thus, we get a $3 \times 3$ system (10) with the matrices as above, which is parameterized by the hypergeometric functions similar to system (4).

To obtain a $2 \times 2$ system one needs to fix the parameter of middle convolution equal to one of the eigenvalues of $A_\infty$ of the initial system (6). Take $\mu = -2(\theta_0 + \theta_1)$. The resulting system (10) has the following residue matrices:

$$\widetilde{B}_1 = \begin{pmatrix} -2\theta_1 & 0 \\ -2\theta_1 & 0 \end{pmatrix}, \qquad \widetilde{B}_2 = \begin{pmatrix} -2\theta_0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_3 = \begin{pmatrix} 0 & 0 \\ -\frac{2(\theta_1 u_0(t) + (\theta_1 + \theta_t)u_1(t))}{u_t(t)} & -2(\theta_0 + \theta_1 + \theta_t) \end{pmatrix}.$$

Here the quotient space $\mathbb{C}^2 \simeq \mathbb{C}^6/(\mathcal{K} + \mathcal{L})$ is constructed by adding $e_3$, $e_5$ to the basis of the invariant subspaces.

To reduce this system to the initial form (6), we need to apply the gauge transformation with the matrix

$$D = \begin{pmatrix} \frac{u_0(t)+u_1(t)}{u_1(t)} & 0 \\ 1 & f(t) \end{pmatrix},$$

with the function $f(t)$ to be determined. Thus, we have a new Fuchsian system with four singularities and the residue matrices

$$D^{-1}\widetilde{B}_1 D = \begin{pmatrix} -2\theta_1 & 0 \\ -\frac{2\theta_1 u_0(t)}{f(t)u_1(t)} & 0 \end{pmatrix}, \qquad D^{-1}\widetilde{B}_2 D = \begin{pmatrix} -2\theta_0 & 0 \\ \frac{2\theta_0}{f(t)} & 0 \end{pmatrix},$$

$$D^{-1}\widetilde{B}_3 D = \begin{pmatrix} 0 & 0 \\ \frac{2\theta_1 u_0(t) - 2\theta_0 u_1(t)}{f(t)u_1(t)} & -2(\theta_0+\theta_1+\theta_t) \end{pmatrix}.$$

This system is already of the type as the initial system. Therefore, one can check directly the validity of the following statement.

**Theorem 1.** *Let $u_0(t)$, $u_1(t)$ and $u_t(t)$ with $u_0(t)+u_1(t)+u_t(t)=0$ satisfy (1) with*

$$a_0 = 2\theta_t, \qquad b_0 = 2(\theta_0+\theta_1+\theta_t), \qquad c_0 = 2(\theta_0+\theta_t)+1;$$
$$a_1 = 2\theta_t, \qquad b_1 = 2(\theta_0+\theta_1+\theta_t), \qquad c_1 = 2(\theta_0+\theta_t);$$
$$a_t = 2\theta_t+1, \qquad b_t = 2(\theta_0+\theta_1+\theta_t), \qquad c_t = 2(\theta_0+\theta_t)+1.$$

*Then the operations of addition and middle convolution with parameter $\mu = -2(\theta_0+\theta_1)$ applied to system (4) give new solutions of (1) given by*

$$\widetilde{u}_0(t) = -\frac{2\theta_1 u_0(t)}{f(t)u_1(t)}, \qquad \widetilde{u}_1(t) = \frac{2\theta_0}{f(t)}, \qquad \widetilde{u}_t(t) = \frac{2\theta_1 u_0(t) - 2\theta_0 u_1(t)}{f(t)u_1(t)},$$

*where*

$$f'(t) = 2f(t)\frac{\theta_1 u_0(t) + (\theta_1+\theta_t)u_1(t)}{u_1(t)(t-1)},$$

*and the parameters are given by*

$$\widetilde{a}_0 = 2(\theta_0+\theta_1+\theta_t), \qquad \widetilde{b}_0 = 2\theta_t, \qquad \widetilde{c}_0 = 2(\theta_0+\theta_t)+1;$$
$$\widetilde{a}_1 = 2(\theta_0+\theta_1+\theta_t), \qquad \widetilde{b}_1 = 2\theta_t, \qquad \widetilde{c}_1 = 2(\theta_0+\theta_t);$$
$$\widetilde{a}_t = 2(\theta_0+\theta_1+\theta_t)+1, \qquad \widetilde{b}_t = 2\theta_t, \qquad \widetilde{c}_t = 2(\theta_0+\theta_t)+1.$$

**Example.** Take $\theta_0 = \theta_1 = \theta_t = 1/3$ and let

$$u_0(t) = \frac{(-1)^{2/3}(t+1)c_1}{(1-t)^{1/3}t^{4/3}}$$

be a particular solution of the hypergeometric equation (1). The functions $u_1(t)$ and $u_t(t)$ can be found from the Schlesinger system (5) and condition that the sum of these three functions is zero. The function $f(t)$ can be calculated explicitly by

$$f(t) = \frac{(t-1)^{4/3}t^{1/3}c_2}{t-2}.$$

Here, $c_1, c_2$ are arbitrary constants. Thus, we get a new triple of functions

$$\widetilde{u}_0(t) = \frac{2(t+1)}{3(t-1)^{1/3}t^{4/3}c_2}, \qquad \widetilde{u}_1(t) = \frac{2(t-2)}{3(t-1)^{4/3}t^{1/3}c_2}, \qquad \widetilde{u}_t(t) = \frac{2-4t(t-1)}{3(t-1)^{4/3}t^{4/3}c_2}$$

which are the hypergeometric functions with new values of the parameters.

Let us consider another choice of the parameter $\mu$ in middle convolution which also leads to $2 \times 2$ system, namely $\mu = 2\theta_t$. In this case the resulting Fuchsian system (10) has the following residue matrices:

$$\widetilde{B}_1 = \begin{pmatrix} 2(\theta_0 + \theta_t) & 2\theta_1 \\ 0 & 0 \end{pmatrix}, \qquad \widetilde{B}_2 = \begin{pmatrix} 0 & 0 \\ 2\theta_0 & 2(\theta_1 + \theta_t) \end{pmatrix}, \qquad \widetilde{B}_3 = O_2,$$

where $O_2$ is a $2 \times 2$ zero matrix. Applying the gauge transformation

$$D = \begin{pmatrix} -\frac{\theta_1}{\theta_0} & 1 \\ 1 & 1 \end{pmatrix}$$

we can write the scalar differential equation for the first component $\psi_3^1$ of the solution vector $\Psi_3 = \left(\psi_3^1, \psi_3^2\right)^{tr}$ of the resulting system. This gives the hypergeometric equation with parameters $a = -2\theta_t, b = 1 - 2(\theta_0 + \theta_1 + \theta_t), c = 1 - 2(\theta_0 + \theta_t)$.

**Case 2.** This case is similar to case 1. We start with the matrices

$$A_1 = \begin{pmatrix} 2\theta_0 & 0 \\ u_0(t) & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ u_1(t) & -2\theta_1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ u_t(t) & -2\theta_t \end{pmatrix}$$

which are obtained from (4) by addition $\Psi = \lambda^{-\theta_0}(\lambda - 1)^{\theta_1}(\lambda - t)^{\theta_t}\Psi_1$. Middle convolution with $\mu = -2\theta_0$ shows that the first residue matrix is equal to $2 \times 2$ zero matrix and two other matrices in (10) are constant with respect to $t$ after a gauge transformation. Thus, we have the case similar to the case above.

Fixing $\mu = 2(\theta_1 + \theta_t)$, we have a statement similar to theorem 1.

**Theorem 2.** *Let $u_0(t)$, $u_1(t)$ and $u_t(t)$ be as in theorem 1. Then the operations of addition and middle convolution with parameter $\mu = 2(\theta_1 + \theta_t)$ applied to system (4) give new solutions of (1) given by*

$$\widetilde{u}_0(t) = -\frac{2(\theta_0 + \theta_1 + \theta_t)}{f(t)}, \qquad \widetilde{u}_1(t) = \frac{2\theta_t(\theta_0 + \theta_1 + \theta_t)u_1(t)}{f(t)(\theta_1 u_0(t) + (\theta_1 + \theta_t)u_1(t))},$$

$$\widetilde{u}_t(t) = \frac{2\theta_1(\theta_0 + \theta_1 + \theta_t)(u_0(t) + u_1(t))}{f(t)(\theta_1 u_0(t) + (\theta_1 + \theta_t)u_1(t))},$$

*where*

$$f'(t) = 2\theta_1 f(t)\frac{(\theta_0 + \theta_t)u_0(t) + \theta_0 u_1(t)}{t(\theta_1 u_0(t) + (\theta_1 + \theta_t)u_1(t))}$$

*and the parameters are given by*

$$\begin{aligned} \widetilde{a}_0 &= -2\theta_1, & \widetilde{b}_0 &= 2\theta_0, & \widetilde{c}_0 &= 2(\theta_0 + \theta_t) + 1; \\ \widetilde{a}_1 &= -2\theta_1, & \widetilde{b}_1 &= 2\theta_0, & \widetilde{c}_1 &= 2(\theta_0 + \theta_t); \\ \widetilde{a}_t &= 1 - 2\theta_1, & \widetilde{b}_t &= 2\theta_0, & \widetilde{c}_t &= 2(\theta_0 + \theta_t) + 1. \end{aligned}$$

We remark again that the result depends on the choice of the isomorphism between $\mathbb{C}^2$ and $\mathbb{C}^6/(\mathcal{K} + \mathcal{L})$ so by another choice of the basis one might get different formulae. Here we added $e_1$ and $e_3$ to the basis.

Example above gives $f(t) = c_2 t^{4/3}/(1 - 4t + t^2)$ and a new triple of functions

$$\widetilde{u}_0(t) = -\frac{2(1 + t(t-4))}{c_2 t^{4/3}}, \qquad \widetilde{u}_1(t) = \frac{2(t-2)}{c_2 t^{1/3}}, \qquad \widetilde{u}_t(t) = \frac{2(1-2t)}{c_2 t^{4/3}}$$

which are the hypergeometric functions with $(\widetilde{a}, \widetilde{b}, \widetilde{c})$ given by $(2/3, 2/3, 7/3)$, $(-2/3, 2/3, 4/3)$ and $(1/3, 2/3, 7/3)$ respectively (compare with initial hypergeometric functions

$u_0(t), u_1(t)$ and $u_t(t)$ with the values $(a, b, c)$ given by $(2/3, 2, 7/3)$, $(2/3, 2, 4/3)$ and $(5/3, 2, 7/3)$, respectively).

**Case 3.** By the operation of addition $\Psi = \lambda^{\theta_0}(\lambda-1)^{\theta_1}(\lambda-t)^{\theta_t}\Psi_1$, we shift the eigenvalues of the system (4) in such a way that the residue matrices of (6) are given by

$$A_1 = \begin{pmatrix} 0 & 0 \\ u_0(t) & -2\theta_0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 \\ u_1(t) & -2\theta_1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 0 & 0 \\ u_t(t) & -2\theta_t \end{pmatrix}.$$

Note that by this transformation we basically remove the first component of vector $\Psi_1$ from the consideration in middle convolution. Applying $mc_\mu$ with $\mu = 2(\theta_0+\theta_1+\theta_t)$, we get the resulting residue matrices of the form

$$\widetilde{B}_1 = \begin{pmatrix} 2\theta_1 & \frac{2\theta_1 u_t(t)}{u_1(t)} \\ \frac{2\theta_t u_1(t)}{u_t(t)} & 2\theta_t \end{pmatrix}, \qquad \widetilde{B}_2 = \begin{pmatrix} 2(\theta_0+\theta_t) & -\frac{2\theta_1 u_t(t)}{u_1(t)} \\ 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_3 = \begin{pmatrix} 0 & 0 \\ \frac{2\theta_t u_1(t)}{u_0(t)+u_1(t)} & 2(\theta_0+\theta_1) \end{pmatrix}$$

(with the choice of vectors $e_3$ and $e_5$). Using the gauge transformation

$$D = \begin{pmatrix} \frac{u_0(t)+u_1(t)}{u_1(t)} & -\frac{\theta_1(u_0(t)+u_1(t))}{\theta_t u_1(t)} \\ 1 & 1 \end{pmatrix}$$

we observe that the residue matrices become independent of the variable $t$, namely

$$D^{-1}\widetilde{B}_1 D = \begin{pmatrix} 0 & 0 \\ 0 & 2(\theta_1+\theta_t) \end{pmatrix}, \qquad D^{-1}\widetilde{B}_2 D = \begin{pmatrix} \frac{2\theta_t(\theta_0+\theta_1+\theta_t)}{\theta_1+\theta_t} & -\frac{2\theta_0\theta_1}{\theta_1+\theta_t} \\ -\frac{2\theta_t(\theta_0+\theta_1+\theta_t)}{\theta_1+\theta_t} & \frac{2\theta_0\theta_1}{\theta_1+\theta_t} \end{pmatrix},$$

$$D^{-1}\widetilde{B}_3 D = \begin{pmatrix} \frac{2\theta_1(\theta_0+\theta_1+\theta_t)}{\theta_1+\theta_t} & \frac{2\theta_0\theta_1}{\theta_1+\theta_t} \\ \frac{2\theta_t(\theta_0+\theta_1+\theta_t)}{\theta_1+\theta_t} & \frac{2\theta_0\theta_t}{\theta_1+\theta_t} \end{pmatrix}.$$

Thus, we can write the equation for the first component of the solution vector $\Psi_3 = \left(\psi_3^1, \psi_3^2\right)^{tr}$. It appears to be the Heun equation where one can choose the parameters

$$\alpha = -2(\theta_0+\theta_1+\theta_t), \qquad \beta = 1-2(\theta_0+\theta_1+\theta_t),$$
$$\gamma = -2(\theta_1+\theta_t), \qquad \delta = 1-2(\theta_0+\theta_t),$$
$$q = 4(\theta_0+\theta_1+\theta_t)(\theta_1+\theta_t t).$$

Here, the Heun equation is given by

$$y'' + \left(\frac{\gamma}{\lambda} + \frac{\delta}{\lambda-1} + \frac{\epsilon}{\lambda-t}\right)y' + \frac{\alpha\beta\lambda - q}{\lambda(\lambda-1)(\lambda-t)}y = 0,$$

where $' = \mathrm{d}/\mathrm{d}\lambda$ and $\epsilon = 1+\alpha+\beta-\gamma-\delta$.

**Case 4.** Finally, we consider addition $\Psi = \lambda^{-\theta_0}(\lambda-1)^{-\theta_1}(\lambda-t)^{-\theta_t}\Psi_1$ so that to shift the eigenvalues of the system (4) in such a way that the residue matrices of (6) are given by

$$A_1 = \begin{pmatrix} 2\theta_0 & 0 \\ u_0(t) & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2\theta_1 & 0 \\ u_1(t) & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 2\theta_t & 0 \\ u_t(t) & 0 \end{pmatrix}.$$

Note that the first component of vector $\Psi = (\psi^1, \psi^2)^{\mathrm{tr}}$ in (4) is given by $\lambda^{\theta_0}(\lambda-1)^{\theta_1}(\lambda-t)^{\theta_t}$ up to a constant and by this addition we remove the second component from the consideration in the middle convolution transformation. Thus, we expect to get a rigid system. An interesting case is to consider middle convolution with the generic value of the parameter $\mu$. We get system (10) with

$$\widetilde{B}_1 = \begin{pmatrix} 2\theta_0+\mu & 2\theta_1 & 2\theta_t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{B}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 2\theta_0 & 2\theta_1+\mu & 2\theta_t \\ 0 & 0 & 0 \end{pmatrix},$$

$$\widetilde{B}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2\theta_0 & 2\theta_1 & 2\theta_t+\mu \end{pmatrix}.$$

Here, the vectors $e_1, e_3$ and $e_5$ are added to the basis of invariant subspaces. One can write down the scalar equation for the first component of solution $\Psi_3 = \left(\psi_3^1, \psi_3^2, \psi_3^3\right)^{\mathrm{tr}}$ of the resulting system and it appears to be [9] the Jordan–Pochhammer equation of order 3 with parameters $\rho = \mu - 2, a_1 = 2\theta_0, a_2 = 1 + 2\theta_1, a_3 = 1 + 2\theta_t$ and singularities $t_1 = 0$, $t_2 = 1, t_3 = t$.

The definition of the Jordan–Pochhammer equation is as follows [16]. Let $p(\lambda)$ and $q(\lambda)$ be the polynomials of degree $n$ and $n-1$ respectively defined by

$$p(\lambda) = \prod_{j=1}^{n}(\lambda - t_j), \qquad (\lambda_i \neq \lambda_j, i \neq j),$$

$$\frac{q(\lambda)}{p(\lambda)} = \sum_{j=1}^{n}\frac{a_j}{\lambda - t_j},$$

where $a_j$ are constants. Then the Jordan–Pochhammer equation is expressed in the form

$$p(\lambda)y^{(n)} = p_{n-1}(\lambda)y^{(n-1)} + \cdots + p_1(\lambda)y' + p_0(\lambda)y,$$

where

$$p_{n-l}(\lambda) = (-1)^{l-1}\left(\binom{\rho+l-1}{l}p_n^{(l)}(\lambda) + \binom{\rho+l-1}{l-1}q_{n-1}^{(l-1)}(\lambda)\right), \qquad l = 1, 2, \ldots, n,$$

$\rho$ being a parameter. The Jordan–Pochhamer equation is a Fuchsian equation without accessory parameters and when $n = 2$ it reduces to the hypergeometric equation with $t_1 = 0, t_2 = 1$ and $\rho, a_1, a_2$ expressed in terms of the parameters $a, b, c$ of (1).

If one fixes the value of the parameter of middle convolution to $\mu = -2(\theta_0 + \theta_1 + \theta_t)$, then similarly to case 3 one can get the Heun equation for the first component of the vector $\Psi_3$. The second component can be considered by analogy.

## 4. Conclusions

We have studied various compositions of addition and middle convolution transformations of system (4) which is related to the hypergeometric equation via isomonodromy deformations. We have investigated the cases when the resulting system (10) is either of the same type and we can get transformations described in theorems 1 and 2 or when it is related to other famous scalar differential equations. The results presented in this paper might be useful in a variety of mathematical and physical problems in which hypergeometric functions appear. The construction of the analogue of middle convolution for systems with irregular singularities is an open problem and this will lead to nontrivial relations between special functions of the confluent type.

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